

AN EXTENSION OF BRAHMAGUPTA'S THEOREM

Dissertation

SUBMITTED TO THE BOARD OF UNIVERSITY STUDIES OF THE JOHNS HOPKINS
UNIVERSITY IN CONFORMITY WITH THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

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A. W. RICHESON

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An Extension of Brahmagupta's Theorem.

BY A. W. RICHESON.

The question of cyclic polygons early attracted the attention of mathematicians. As early as 625 A. D. we find Brahmagupta, a Hindoo mathematician, working with the cyclic quadrangle.* About 1782 Lhuillier gave a formula for the radius of the circumscribed circle of an inscribed quadrangle in terms of the four sides, for both the convex and the crossed cases.† It is the purpose of this paper to study first the inscribed quadrangle and then to consider other polygons, and in particular the inscribed pentagon.

1. *Distances under Inversions.* Two points P and P' collinear with a given point O are said to be inverse points with respect to the point O if the product

$$\overline{OP} \cdot \overline{OP'} = K$$

where K is a real constant, O is the center, and K the constant of inversion. Given the inverse points $P_0, P_1, P_2 \dots$ in the plane, let us denote by λ_i the absolute distances of the points P_i and P_j . Under an inversion with respect to the point O as a center the distance λ_{12} becomes

$$K\lambda'_{12}/\lambda'_{01}\lambda'_{02}.$$

Accordingly a function of the distances which contains each suffix the same number of times merely acquires a factor under an inversion, that is to say, it is a relative invariant. Consequently for four points any one of the eight expressions

$$\lambda_{01}^2 \lambda_{23} \lambda_{34} \lambda_{42} \pm \lambda_{02}^2 \lambda_{34} \lambda_{41} \lambda_{13} \pm \lambda_{03}^2 \lambda_{41} \lambda_{12} \lambda_{24} \pm \lambda_{04}^2 \lambda_{12} \lambda_{23} \lambda_{31}$$

is a relative invariant under inversions.

Since equations of the type

$$\mu_1 \overline{OP_1}^2 + \mu_2 \overline{OP_2}^2 + \mu_3 \overline{OP_3}^2 + \dots + K = 0$$

are equations of circles, then

$$\lambda_{p1}^2 \lambda_{23} \lambda_{34} \lambda_{42} \pm \lambda_{p2}^2 \lambda_{34} \lambda_{41} \lambda_{13} \pm \lambda_{p3}^2 \lambda_{41} \lambda_{12} \lambda_{24} \pm \lambda_{p4}^2 \lambda_{12} \lambda_{23} \lambda_{31} = 0,$$

* Kaye, G. R., *Indian Mathematics*, Thacker Spink Co., Calcutta (1915), p. 22.

† Archibald, R. C., "Area of the Quadrilateral," *Bulletin of the Mathematical Association of America*, Vol. 29, p. 32.

where p denotes the variable point and the λ 's the absolute distances from the points, is the equation of a circle.

Attached to four points in the plane there is a system of eight circles obtained by considering the arrangement of the positive and the negative signs in

$$\sum^4 \lambda^2_{p1} \lambda_{23} \lambda_{34} \lambda_{42} \pm = 0.$$

This system of circles may be classified as follows:

- (1) When all the signs are positive, one nullipartite circle. This is the standard case.
- (2) When two signs are positive and two negative, three unipartite circles, the Jacobian circles of the four points.
- (3) When one sign is positive and three negative, four null or point circles.

In a paper in the *Proceedings of the London Mathematical Society*, R. Russell* considers seven of this system, the three unipartite or Jacobian circles and the four null or point circles. However, he failed to take into consideration the nullipartite circle.

I propose to determine the radii of this system of circles, and to consider what happens when the four points are on a circle.

2. *The Standard Case.* The equation of the circle in the standard case is

$$\lambda^2_{p1} \lambda_{23} \lambda_{34} \lambda_{42} + \lambda^2_{p2} \lambda_{34} \lambda_{41} \lambda_{13} + \lambda^2_{p3} \lambda_{41} \lambda_{12} \lambda_{24} + \lambda^2_{p4} \lambda_{12} \lambda_{23} \lambda_{31} = 0$$

where the λ 's denote the absolute distances from the points and p is the variable point. In order to calculate the radius it is convenient to make a slight change in notation. Let x be the variable point p ,

$$\begin{array}{ll} c = \lambda_{12} & e = \lambda_{24} \\ b = \lambda_{13} & f = \lambda_{34} \\ a = \lambda_{23} & d = \lambda_{41} \end{array}$$

Then the expression for the circle becomes

$$|x - x_1|^2 aef + |x - x_2|^2 bdf + |x - x_3|^2 cde + |x - x_4|^2 abc = 0$$

which may be written

$$(1) \quad (x - x_1)(\bar{x} - \bar{x}_1) aef + (x - x_2)(\bar{x} - \bar{x}_2) bdf + (x - x_3)(\bar{x} - \bar{x}_3) cde \\ + (x - x_4)(\bar{x} - \bar{x}_4) abc = 0.$$

* Russell, R., "Geometry on the Quartic," *Proceedings of the London Mathematical Society*, Vol. 19 (1889), p. 56.

Expanding and collecting the terms of (1) we will be able to compute the center and radius of this circle in terms of the six distances a, b, c, d, e and f by equating the coefficients of (1) with those of the self conjugate equation of the circle

$$\alpha x\bar{x} + \bar{a}_1 x + a_1 \bar{x} + \beta = 0,$$

where the center $y = -\bar{a}_1/\alpha$ and the radius

$$R^2 = (a_1 \bar{a}_1 - \alpha \beta) / \alpha^2.$$

In this way we have the center

$$y = (-a_1) / \alpha = (aefx_1 + bfdx_2 + cdex_3 + abcx_4) / (aef + bfd + cde + abc)$$

which recalls the theorem that the centroid of four points is the sum of the products of the four points and their respective weights divided by the weights of the four points.

In a similar manner the radius is found to be

$$(2) \quad R^2 = (a_1 \bar{a}_1 - \alpha \beta) / \alpha^2 \\ = -2abcdef(ad + cf + be) / (aef + bfd + cde + abc)^2.$$

This is the radius of the standard case of the circle and applies with change of sign to any one of the system of eight circles.

3. *The Jacobian Circle.* In order to obtain one of the Jacobian circles from

$$\sum \lambda_{p1}^2 \lambda_{23} \lambda_{34} \lambda_{42} \pm = 0$$

it is necessary to change the sign of one of the λ 's, say λ_{13} , from positive to negative. We may then write

$$(3) \quad \lambda_{p1}^2 \lambda_{23} \lambda_{34} \lambda_{42} - \lambda_{p2}^2 \lambda_{34} \lambda_{41} \lambda_{13} + \lambda_{p3}^2 \lambda_{41} \lambda_{12} \lambda_{24} - \lambda_{p4}^2 \lambda_{12} \lambda_{23} \lambda_{31} = 0.$$

If this circle is on the point 1 then

$$-\lambda_{12}^2 \lambda_{34} \lambda_{41} \lambda_{13} + \lambda_{13}^2 \lambda_{41} \lambda_{12} \lambda_{24} - \lambda_{14}^2 \lambda_{12} \lambda_{23} \lambda_{31} = 0.$$

This says that the point 1 is on the Jacobian circle if

$$-\lambda_{12}\lambda_{34} + \lambda_{13}\lambda_{24} - \lambda_{14}\lambda_{23} = 0.$$

Likewise for the points 2, 3 and 4 the same condition holds true. That is, the Jacobian circle is the circumcircle of the four points in the above order if $-\lambda_{12}\lambda_{34} + \lambda_{13}\lambda_{24} - \lambda_{14}\lambda_{23} = 0$. This is a known condition that four points be on a circle in the order 1, 2, 3, 4.

Since λ_{13} in

$$\sum^4 \lambda_{p1}^2 \lambda_{23} \lambda_{34} \lambda_{42} \pm = 0$$

was replaced by b and its sign changed to obtain the Jacobian circle, we must likewise change the sign of b in the expression (2) above for the radius of the standard case to obtain the radius of the Jacobian circle. Hence we may write

$$(4) \quad R^2 = 2abcdef(ad + cf - be)/(aef - bfd + cde - abc)^2$$

for the radius of the Jacobian circle. But this value of the radius is indeterminate. For the condition that four points be on a circle is that $ad + cf - be = 0$ or that $aef - bfd + cde - abc = 0$. This is only one condition since the one implies the other.

In order to obtain the limit of this indeterminate expression (4) let us consider Cayley's relation connecting the mutual distances of any four points in the plane. In determinant* form it is as follows:

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & \lambda_{12}^2 & \lambda_{13}^2 & \lambda_{14}^2 \\ 1 & \lambda_{21}^2 & 0 & \lambda_{23}^2 & \lambda_{24}^2 \\ 1 & \lambda_{31}^2 & \lambda_{32}^2 & 0 & \lambda_{34}^2 \\ 1 & \lambda_{41}^2 & \lambda_{42}^2 & \lambda_{43}^2 & 0 \end{vmatrix} = 0.$$

Expanding and replacing the λ 's by a, b, c, d, e and f we may write this in the form

$$\begin{aligned} & a^2 d^2 (d^2 - e^2 - f^2) + a^2 d^2 (a^2 - b^2 - c^2) + b^2 e^2 (e^2 - d^2 - f^2) \\ & + b^2 e^2 (b^2 - c^2 - a^2) + c^2 f^2 (f^2 - e^2 - d^2) + c^2 f^2 (c^2 - a^2 - b^2) \\ & + a^2 e^2 f^2 + b^2 f^2 d^2 + c^2 d^2 e^2 + a^2 b^2 c^2 = 0. \end{aligned}$$

Now it is possible to express the left hand side in the form

$$(ad + cf + be)\{2ad(a^2 + d^2) + 2be(b^2 + e^2) + 2cf(c^2 + f^2) - (ad + cf + be) \sum^6 a^2\} + (aef + bfd + cde + abc)^2$$

which is of the form $g_3^2 x + g_2 y$ where $g_2 = ad + cf + be$ and $g_3 = aef + bfd + cde + abc$. From this we see that when $g_2 = 0$,

$$g_2/g_3^2 = -1/[2ad(a^2 + d^2) + 2be(b^2 + e^2) + 2cf(c^2 + f^2)]$$

and (2) becomes

$$R^2 = abcdef / \sum^3 ad(a^2 + d^2).$$

Changing the sign of b , we have for the points 1, 2, 3, 4 on a circle (in this order)

* Cayley's *Collected Works*, Vol. I, p. 3; Salmon, *Conic Sections*, p. 134.

$$R^2 = -abcdef/[ad(a^2 + d^2) - be(b^2 + e^2) + cf(c^2 + f^2)].$$

The two diagonals b and e may be expressed in terms of the other four distances* as

$$b^2 = (ad + cf)(af + cd)/(ac + df)$$

and

$$e^2 = (ad + cf)(ac + df)/(af + cd).$$

When these two relations are substituted in the expression for the radius above, it readily reduces to *

$$(4') \quad R^2 = (ac + df)(ad + cf)(af + cd)/(s - a)(s - c)(s - d)(s - f),$$

where $2s = a + c + d + f$. This gives the radius of the circumcircle of the four points in terms of the four distances a, c, d and f when the points are in the order x_1, x_2, x_3, x_4 .

In order to obtain an expression for the radius of the Jacobian circle when the points are in the order x_1, x_2, x_4, x_3 , that is when the quadrangle formed by the four distances a, c, d and f is crossed, it is necessary to change the sign of one of the sides, say f , in equation (4'). The radius then reduces to

$$R^2 = (ac - df)(ad - cf)(cd - af)/s(s - a - f)(s - c - f)(s - d - f),$$

where $2s = a + c + d + f$.

4. *The Inscribed Hexagon.* In the above sections the radius of the circumcircle of the quadrangle was calculated in terms of the four distances connecting the points for two cases, the convex quadrangle and the crossed quadrangle. In this and the following sections we shall consider other inscribed polygons, in particular the hexagon and the pentagon. For the hexagon we shall take the convex hexagon as the standard case.

We may think of the hexagon as made up of two quadrangles with a common side; i. e. the two quadrangles $abcx$ and $defx$ where x is the common side. From section (3) we have two equations for the circumcircle of the two quadrangles

$$(5) \quad R^2 = \frac{(ab + cx)(ac + bx)(ax + bc)}{(a + b + c - x)(b + c + x - a)(c + x + a - b)(x + a + b - c)}$$

and

$$(6) \quad R^2 = \frac{(de + fx)(df + ex)(dx + ef)}{(d + e + f - x)(e + f + x - d)(f + x + d - e)(x + d + e - f)}$$

These two equations must coexist if the hexagon is inscribed in the circle. We may further write the equality

* Hobson, *Plane Trigonometry* (Fifth Edition), p. 206.

$$(A) \quad \frac{(ab+cx)(ac+bx)(ax+bc)}{(a+b+c-x)(b+c+x-a)(c+x+a-b)(x+a+b-c)} \\ = \frac{(de+fx)(df+ex)(dx+ef)}{(d+e+f-x)(e+f+x-d)(f+x+d-e)(x+d+e-f)}$$

This is an equation of the seventh degree in x and on eliminating x from equations (5) and (6) above we will obtain an equation of the seventh degree in R^2 of the type

$$(7) \quad \lambda_0 \alpha_0 R^{14} + \lambda_1 \alpha_1 R^{12} + \lambda_2 \alpha_2 R^{10} + \lambda_3 \alpha_3 R^8 + \lambda_4 \alpha_4 R^6 \\ + \lambda_5 \alpha_5 R^4 + \lambda_6 \alpha_6 R^2 + \lambda_7 \alpha_7 = 0$$

where the λ 's are numerical constants and the α 's are some function of the sides a, b, c, d, e, f . Simplifying the two equations and clearing of fractions we have

$$(8) \quad R^2 x^4 + abcx^3 + [a^2 b^2 + a^2 c^2 + b^2 c^2 - 2R^2(a^2 + b^2 + c^2)]x^2 \\ + abc(a^2 b^2 c^2 - 8R^2)x + [a^2 b^2 c^2 - 2R^2(a^2 b^2 + a^2 c^2 + b^2 c^2) \\ + R^2(d^4 + e^4 + f^4)] = 0.$$

and

$$(8') \quad R^2 x^4 + defx^3 + [d^2 e^2 + d^2 f^2 + e^2 f^2 - 2R^2(d^2 + e^2 + f^2)]x^2 \\ + def(d^2 e^2 f^2 - 8R^2)x + [d^2 e^2 f^2 - 2R^2(d^2 e^2 + d^2 f^2 + e^2 f^2) \\ + R^2(d^4 + e^4 + f^4)] = 0.$$

If we set

$$\begin{array}{ll} \sigma_3 = abc & s_3 = def \\ \Sigma a^2 b^2 = a^2 b^2 + a^2 c^2 + b^2 c^2 & \Sigma d^2 e^2 = d^2 e^2 + d^2 f^2 + e^2 f^2 \\ \Sigma a^4 = a^4 + b^4 + c^4 & \Sigma d^4 = d^4 + e^4 + f^4 \end{array}$$

equations (8) and (8') can be expressed in the condensed form

$$(9) \quad R^2 x^4 + \sigma_3 x^3 + (\Sigma a^2 b^2 - 2R^2 \Sigma a^2)x^2 \\ + (\sigma_3^3 - 8R^2 \sigma_3)x + (\sigma_3^2 - 2R^2 \Sigma a^2 b^2 + R^2 \Sigma a^4) = 0$$

and

$$(9') \quad R^2 x^4 + s_3 x^3 + (\Sigma d^2 e^2 - 2R^2 \Sigma d^2)x^2 \\ + (s_3^3 - 8R^2 s_3)x + (s_3^2 - 2R^2 \Sigma d^2 e^2 + R^2 \Sigma d^4) = 0$$

and x can be eliminated from equations (9) and (9') by Bezout's method of elimination.* This will give us a determinant of the fourth order and the seventh degree in R^2 . We shall designate this determination by Δ_6 :

* Salmon, *Higher Algebra*, p. 81.

$$\Delta_8 = \left| \begin{array}{l} (s_3 - \sigma_3), \\ (\Sigma d^2 e^2 - 2R^2 \Sigma d^2 - \Sigma a^2 b^2 + 2R^2 \Sigma a^2), \\ (s_3^3 - 8R^2 s_3 - \sigma_3^3 + 8R^2 \sigma_3), \\ (s_3^2 - 2R^2 \Sigma d^2 e^2 + R^2 \Sigma d^4 - \sigma_3^2 + 2R^2 \Sigma a^2 b^2 - R^2 \Sigma a^4), \\ R^2 [\Sigma d^2 e^2 - 2R^2 \Sigma d^2 - \Sigma a^2 b^2 + 2R^2 \Sigma a^2], \\ R^2 [s_3^3 - 8R^2 s_3 - \sigma_3^3 + 8R^2 \sigma_3] \\ \quad + \sigma_3 [\Sigma d^2 e^2 - 2R^2 \Sigma d^2] - s_3 [\Sigma a^2 b^2 - 2R^2 \Sigma a^2], \\ R^2 (s_3^2 - 2R^2 \Sigma d^2 e^2 + R^2 \Sigma d^2 - \sigma_3^2 + 2R^2 \Sigma a^2 b^2 - R^2 \Sigma a^4) \\ \quad + \sigma_3 (s_3^3 - 8R^2 s_3) - s_3 (\sigma_3^3 - 8R^2 \sigma_3), \\ \sigma_3 (s_3^2 - 2R^2 \Sigma d^2 e^2 + R^2 \Sigma d^4) - s_3 (\sigma_3^2 - 2R^2 \Sigma a^2 b^2 + R^2 \Sigma a^4), \\ R^2 (s_3^3 - 8R^2 s_3 - \sigma_3^3 + 8R^2 \sigma_3), \\ R^2 (s_3^2 - 2R^2 \Sigma d^2 e^2 + R^2 \Sigma d^4 - \sigma_3^2 + 2R^2 \Sigma a^2 b^2 - R^2 \Sigma a^4) \\ \quad + \sigma_3 (s_3^3 - 8R^2 s_3) - s_3 (\sigma_3^3 - 8R^2 \sigma_3), \\ \sigma_3 (s_3^2 - 2R^2 \Sigma d^2 e^2 + R^2 \Sigma d^4) - s_3 (\sigma_3^2 - 2R^2 \Sigma a^2 b^2 + R^2 \Sigma a^4) \\ \quad + (\Sigma a^2 b^2 - 2R^2 \Sigma a^2) (s_3^3 - 8R^2 s_3) - (\Sigma d^2 e^2 - 2R^2 \Sigma d^2) (\sigma_3^3 - 8R^2 \sigma_3), \\ (\Sigma a^2 b^2 - 2R^2 \Sigma a^2) (s_3^2 - 2R^2 \Sigma d^2 e^2 + R^2 \Sigma d^4) \\ \quad - (\sigma_3^2 - 2R^2 \Sigma a^2 b^2 + R^2 \Sigma a^4) (\Sigma d^2 e^2 - 2R^2 \Sigma d^2), \\ R^2 (s_3^2 - 2R^2 \Sigma d^2 e^2 + R^2 \Sigma d^4 - \sigma_3^2 + 2R^2 \Sigma a^2 b^2 - R^2 \Sigma a^4) \\ \sigma_3 (s_3^2 - 2R^2 \Sigma d^2 e^2 + R^2 \Sigma d^4) - s_3 (\sigma_3^2 - 2R^2 \Sigma a^2 b^2 + R^2 \Sigma a^4) \\ (\Sigma a^2 b^2 - 2R^2 \Sigma a^2) (s_3^2 - 2R^2 \Sigma d^2 e^2 + R^2 \Sigma d^4) \\ \quad - (\sigma_3^2 - 2R^2 \Sigma a^2 b^2 + R^2 \Sigma a^4) (\Sigma d^2 e^2 - 2R^2 \Sigma d^2) \\ (\sigma_3^3 - 8R^2 \sigma_3) (s_3^2 - 2R^2 \Sigma d^2 e^2 + R^2 \Sigma d^4) \\ \quad - (s_3^3 - 8R^2 s_3) (\sigma_3^2 - 2R^2 \Sigma a^2 b^2 + R^2 \Sigma a^4) \end{array} \right|.$$

Let us prove the following theorem:

THEOREM. *The product, $R_1^2 \cdot R_2^2 \cdot R_3^2 \cdot \dots \cdot R_7^2$, of the seven roots of R^2 given by equation (7) after eliminating x from equations (9) and (9') above is*

$$K \prod_{i=1}^{10} (abc - def) / \prod_{i=1}^{16} (a \pm b \pm c \pm d \pm e \pm f)$$

where K is a numerical constant and with the agreement that there shall be an odd number of negative signs in the factors of the denominator.

Let us examine the two coefficients $\lambda_0 \alpha_0$ and $\lambda_1 \alpha_0$ in equation (7) above. From equations (5) and (6) R^2 can become infinite when $x = a + b + c$ and $x = d + e + f$, that is, when at least one factor of the denominators of both (5) and (6) is zero. Hence R^2 is infinite if $a + b + c = d + e + f$ or $a + b + c - d - e - f = 0$. But since there are four factors in each denominator, for each factor of one there are four factors that go with it. This is the only way in which R^2 can become infinite since R^2 is not infinite when x is infinite. Hence $\lambda_1 \alpha_0 = \lambda_0 \prod_{i=1}^{16} (a \pm b \pm c \pm d \pm e \pm f)$.

Now let us examine $\lambda_7\alpha_7$ in equation (7). From equations (5) and (6) R^2 can become zero if $ab + cx = 0$ and $de + fx = 0$. That is, R^2 will become zero if $abf - dec = 0$; but since there are three factors in each numerator of (5) and (6), for each factor of the one there are three factors that go with it. Hence we have the product

$$\prod^9 (abf - dec).$$

However on examining equation (7) in R^2 , that is

$$\lambda_0 \prod^{16} (a \pm b \pm c \pm d \pm e \pm f) R^{14} + \dots + \lambda_7 \alpha_7 = 0,$$

we find that each term should be of degree thirty, while $\prod^9 (abf - dec)$ is of degree twenty-seven, hence there is a missing factor. If $x = \infty$, R^2 will be zero. That is $(abc - def)$ is zero and the missing factor is found, and we have $\lambda_7 \alpha_7 = \lambda_7 \prod^{10} (abc - def)$; this is of degree thirty which is correct. Equation (7) may be written

$$\lambda_0 \prod^{16} (a \pm b \pm c \pm d \pm e \pm f) R^{14} + \dots + \lambda_7 \prod^{10} (abc - def) = 0.$$

The product of the seven roots of R^2 is given by $\lambda_7 \alpha_7 / \lambda_0 \alpha_0$. Hence we have

$$R_1^2 \cdot R_2^2 \cdot R_3^2 \cdot \dots \cdot R_7^2 = K \prod^{10} (abc - def) / \prod^{16} (a \pm b \pm c \pm d \pm e \pm f)$$

and the theorem is proved.

5. *The Inscribed Pentagon.* The same argument that was applied to the hexagon in the above section may be applied to the pentagon. Instead of two quadrangles with a common side inscribed in a circle we will have one quadrangle and a triangle having a common side to form the pentagon. That is the pentagon $abcde$ is formed by the quadrangle $abcx$ and the triangle dex where x is the common side.

The radius of the circumcircle of the pentagon is given by the equations

$$(10) \quad R^2 = \frac{(ab + cx)(ac + bx)(ax + bc)}{(a + b + c - x)(b + c + x - a)(c + x + a - b)(x + a + b - c)}$$

and

$$(10') \quad R^2 = \frac{d^2 e^2 x^2}{(d + e + x)(e + x - d)(x + d - e)(d + e - x)}$$

These two equations must coexist if the pentagon is inscribed in the circle and likewise an equality exist similar to (A) for the hexagon in the above section. This is an equation of the seventh degree in x and on eliminating x from equations (10) and (10') we will obtain an equation of the seventh degree in R^2 of the type

$$(11) \quad \lambda_0 \alpha_0 R^{14} + \lambda_1 \alpha_1 R^{12} + \lambda_2 \alpha_2 R^{10} + \lambda_3 \alpha_3 R^8 + \cdots + \lambda_7 \alpha_7 = 0,$$

where the λ 's are numerical constants and the α 's are some functions of the sides a, b, c, d, e .

The process of eliminating x from the two equations (10) and (10') is similar to that of the hexagon but since the pentagon may be obtained directly from the hexagon by making one of its sides, say f , zero, it is not necessary to go through the elimination. On making $f = 0$ the determinant Δ_6 reduces to a determinant, which we shall designate by Δ_5 of the same order and degree in R^2 as Δ_6 . The determinant Δ_5 is as follows:

$$\Delta_5 = \begin{vmatrix} -\sigma_3, \\ (\Sigma d^2 e^2 - 2R^2 \Sigma d^2 - \Sigma a^2 b^2 + 2R^2 \Sigma a^2), \\ (-\sigma_3^3 + 8R^2 \sigma_3), \\ (-\sigma_3^2 - 2R^2 \Sigma d^2 e^2 + R^2 \Sigma d^4 + 2R^2 \Sigma a^2 b^2 - R^2 \Sigma a^4), \\ R^2 (\Sigma d^2 e^2 - 2R^2 \Sigma d^2 - \Sigma a^2 b^2 + 2R^2 \Sigma a^2), \\ R^2 (-\sigma_3^2 + 8R^2 \sigma_3) + \sigma_3 (\Sigma d^2 e^2 - 2R^2 \Sigma d^2), \\ R^2 (-2R^2 \Sigma d^2 e^2 + R^2 \Sigma d^4 - \sigma_3^2 + 2R^2 \Sigma a^2 b^2 - R^2 \Sigma a^4), \\ \sigma_3 (-2R^2 \Sigma d^2 e^2 + R^2 \Sigma d^4), \\ R^2 (-\sigma_3^3 + 8R^2 \sigma_3), \\ R^2 (-2R^2 \Sigma d^2 e^2 + R^2 \Sigma d^4 - \sigma_3^2 + 2R^2 \Sigma a^2 b^2 - R^2 \Sigma a^4), \\ \sigma_3 (-2R^2 \Sigma d^2 e^2 + R^2 \Sigma d^4) - (\Sigma d^2 e^2 - 2R^2 \Sigma d^2) (\sigma_3^3 - 8R^2 \sigma_3), \\ (\Sigma a^2 b^2 - 2R^2 \Sigma a^2) (-2R^2 \Sigma d^2 e^2 + R^2 \Sigma d^4) \\ \quad - (\sigma_3^2 - 2R^2 \Sigma a^2 b^2 + R^2 \Sigma a^4) (\Sigma d^2 e^2 - 2R^2 \Sigma d^2), \\ R^2 (-R^2 \Sigma d^2 e^2 + R^2 \Sigma d^4 - \sigma_3^2 + 2R^2 \Sigma a^2 b^2 - R^2 \Sigma a^4) \\ \sigma_3 (-2R^2 \Sigma d^2 e^2 + R^2 \Sigma d^4) \\ (\Sigma a^2 b^2 - 2R^2 \Sigma a^2) (-2R^2 \Sigma d^2 e^2 + R^2 \Sigma d^4) \\ \quad - (\sigma_3^2 - 2R^2 \Sigma a^2 b^2 + R^2 \Sigma a^4) (\Sigma d^2 e^2 - 2R^2 \Sigma d^2) \\ (\sigma_3^3 - 8R^2 \sigma_3) (-2R^2 \Sigma d^2 e^2 + R^2 \Sigma d^4) \end{vmatrix}$$

Let us prove a corresponding theorem for the pentagon as was proved for the hexagon.

THEOREM. *The product, $R_1^2 \cdot R_2^2 \cdot R_3^2 \cdot \cdots \cdot R_7^2$, of the seven roots of R^2 , given by equations (11) after eliminating x from equations (10) and (10'), is*

$$K(abcde)^9 / \prod^{16} (a \pm b \pm c \pm d \pm e),$$

where K is a numerical constant.

We shall examine the two coefficients $\lambda_0 \alpha_0$ and $\lambda_7 \alpha_7$ in equation (11). From equations (10) and (10'), R^2 can become infinite when $x = a + b + c$ and $x = d + e$; that is, when at least one of the factors of the denominators

of (10) and (10') are zero. Hence $a + b + c - d - e = 0$. But since there are four factors in each denominator, for each factor of the one there are four factors that go with it. This is the only way in which R^2 can become infinite since R^2 is not infinite when x is infinite. Hence

$$\lambda_0 \alpha_0 = \lambda_0 \prod_{i=1}^{16} (a \pm b \pm c \pm d \pm e).$$

Now let us examine $\lambda_7 \alpha_7$. From equations (10) and (10') R^2 can become zero if $ab + cx = 0$ and either $d = 0$ or $e = 0$. That is to say $\lambda_7 \alpha_7 = \lambda_7 (abcde)^\mu$, where μ must be determined by examining equation (11) in R^2 . We find that each term is of degree thirty and μ must be 6. Equation (11) may be written

$$\lambda_0 \prod_{i=1}^{16} (a \pm b \pm c \pm d \pm e) R^{14} + \dots + \lambda_7 (abcde)^6 = 0.$$

Since the product of the seven roots of R^2 is given by $\lambda_7 \alpha_7 / \lambda_0 \alpha_0$ we have

$$R_1^2 \cdot R_2^2 \cdot R_3^2 \cdot \dots \cdot R_7^2 = K (abcde)^6 / \prod_{i=1}^{16} (a \pm b \pm c \pm d \pm e).$$

To determine the constant K we will consider the special case of the regular convex pentagon where $a = b = c = d = e$. The equations giving the radius of the circumscribed circle are

$$(12) \quad R^2 = a^3(x + a)^3 / (x + a)^3(3a - x) = a^3 / (3a - x)$$

or $x = -a$ and

$$(12') \quad R^2 = a^4 x^2 / (2a + x)(2a - x)x^2 = a^4 / (4a^2 - x)$$

or $x = 0$. From (12) above

$$x = (3aR^2 - a^3) / R^2.$$

Substituting this value of x in (12') we obtain an equation of the second degree in R^2

$$(13) \quad \begin{aligned} 5R^4 - 5a^2R^2 + a^4 &= 0. \\ \therefore R^2 &= [5 \pm (5)^{1/2}a^2] / 10. \end{aligned}$$

It should be noticed above that for $x = 0$ in (12') that

$$(13') \quad R^2 = a^3 / 3$$

five times, since x was taken out of the two equations (12) and (12') five times as a factor. Equation (13) gives two values of R^2 while (13') gives five values which make up the total of seven values as shown by the general equation in R^2 .

To determine the constant K required in the expression for the product of the seven roots of R^2 we have only to multiply these seven values together

$$R_1^2 \cdot R_2^2 \cdot R_3^2 \cdot \dots \cdot R_7^2 = (1/5) (1/3)^5 a^{14}.$$

Hence $K = (1/5)(1/3)^5$ and the expression for the product of the seven roots of R^2 is

$$(1/5)(1/3)^5 (abcde)^6 / \prod^{16} (a \pm b \pm c \pm d \pm e),$$

and the theorem is proved.

6. *The equation of the Seventh Degree in R^2 .* To consider the general case of the pentagon where no two of the sides are equal we will center our attention on the determinant $\Delta_5 = 0$ obtained by eliminating x from equations (10) and (10') in sec. 5 above. Upon expansion this determinant gives an equation of the type

$$(14) \quad \lambda_0 \alpha_0 R^{14} + \lambda_1 \alpha_1 R^{12} + \lambda_2 \alpha_2 R^{10} + \lambda_3 \alpha_3 R^8 \\ + \lambda_4 \alpha_4 R^6 + \lambda_5 \alpha_5 R^4 + \lambda_6 \alpha_6 R^2 + \lambda_7 \alpha_7 = 0,$$

where the λ 's are constants and the α 's are some function of the sides a, b, c, d, e . Instead of expanding $\Delta_5 = 0$ in full let us examine several of the coefficients of equation (14). We have already found in sec. 5 that

$$\lambda_0 \alpha_0 = \lambda_0 \prod^{16} (a \pm b \pm c \pm d \pm e) \quad \text{and} \quad \lambda_7 \alpha_7 = (abcde)^6.$$

The coefficients $\alpha_4, \alpha_5, \alpha_6$ can be obtained in terms of the symmetric functions of the squares of the sides a, b, c, d, e of the pentagon by a partial expansion of Δ_5 . Thus, if we set

$$S_1 = \sum^5 a^2, S_2 = \sum^{10} a^2 b^2, S_3 = \sum^{10} a^2 b^2 c^2, S_4 = \sum^5 a^2 b^2 c^2 d^2, S_5 = a^2 b^2 c^2 d^2 e^2,$$

these coefficients are as follows:

$$\alpha_7 = S_5^3, \alpha_6 = S_5^2 (2S_4 + 2S_1 S_3 + 2S_2^2), \alpha_5 = S_5 (4S_5 S_2 S_1 + S_4 S_3 S_1 - 4S_4^2), \\ \alpha_4 = 2S_5^2 S_2 + S_5 S_4 S_3 + S_3^4 + 2S_4^2 S_1^4 - S_4^2 S_2^2 - 4S_4^3 - 2S_3^2 S_4 S_1^2.$$

Substituting these values for the α 's we may write equation (14) as

$$\prod^{16} (a \pm b \pm c \pm d \pm e) R^{14} + \dots \\ + (2S_5^2 S_2 + S_5 S_4 S_3 + S_3^4 + 2S_4^2 S_1^4 - S_4^2 S_2^2 - 4S_4^3 - 2S_3^2 S_4 S_1^2) R^6 \\ + S_5 (4S_5 S_2 S_1 + S_4 S_3 S_1 - 4S_4^2) R^4 \\ + S_5^2 (2S_4 + 2S_1 S_3 + 2S_2^2) R^2 + S_5^3 = 0.$$

If one of the sides, say e , of the pentagon is made zero then S_5 which is the product of the squares of the five sides becomes zero. Hence each of the coefficients α_7, α_6 and α_5 becomes zero and α_4 , that is the coefficient of R^6 , reduces to

$$\left\{ \prod^3 (a^2 b^2 - c^2 d^2) \right\}^2.$$

The remaining coefficients α_2 and α_3 are known in terms of S_1, S_2, S_3 and S_4 modulo S_5 . The general equation of the pentagon reduces to

$$\prod_{i=1}^{16} (a \pm b \pm c \pm d) R^{14} + \cdots \{ \prod_{i=1}^8 (a^2 b^2 - c^2 d^2) \}^2 R^6 = 0,$$

which is

$$[\{ \prod_{i=1}^4 (a \pm b \pm c \pm d) R^2 - \prod_{i=1}^3 (ab + cd) \} \{ \prod_{i=1}^4 (a \pm b \pm c \pm d) R^2 - \prod_{i=1}^3 (ab - cd) \}] = 0$$

The first factor of this product when set equal to zero is the equation for the radius of the circumcircle of a convex quadrangle with sides a, b, c, d and the second factor when set equal to zero is likewise the equation for the radius of the circumcircle of a crossed quadrangle with the same sides a, b, c, d .

7. *The Seven Circumcircles.* We have already seen that the general equation for the pentagon obtained by eliminating x from equations (10) and (10') is of the seventh degree in R^2 . That is to say with five rods a, b, c, d, e , which when jointed form a convex pentagon inscribed in a circle we may be able to obtain six other pentagons by forming all possible crossed pentagons with the same five rods. To obtain the seven pentagons and their circumcircles it is assumed that the five rods a, b, c, d, e are of lengths $a, a + \alpha, a + \beta, a + \gamma, a + \delta$ respectively where α, β, γ and δ are small compared with a and $\alpha \neq \beta \neq \gamma \neq \delta$. Let us consider some facts concerning these pentagons and their circumcircles. These circles are drawn with the sides of the pentagons in their order of length and we obtain the convex pentagon, two pentagons with one crossing similar to Fig. 1, three pentagons with two crossings similar to Fig. 2, and one with five crossings similar to Fig. 3. Of the seven circumcircles no two of them have equal radii. Further if we change the order of the sides we will obtain the same number of pentagons inscribed in the same circles with the same radii $R_1, R_2, \dots, R_7$. However, there may be a difference in the number of pentagons with one and two crossings. It is also evident that the maximum number of crossings for the pentagon is five while it is impossible to obtain three or four crossings.

8. *The Inscribed $(2n + 1)$ -gon.* Let us consider the general case of the inscribed polygon of $(2n + 1)$ sides. The $(2n + 1)$ -gon may be built up of two polygons of $(n + 1)$ and $(n + 2)$ sides having one side, say x , in common. Then two equations for the radius of the circumcircle of the type of (10) and (10') of sec. 5 can be written which must coexist if the polygons are inscribed in a circle. Eliminating x the common side from these two equations we obtain an equation in R^2 of degree

$$\frac{1}{2} (2n + 1) \binom{2n}{n} - 2^{2n-1}.*$$

* Dr. F. Morley, Lectures at the Johns Hopkins University, 1927-28.

This expression gives the number of polygons formed by $(2n + 1)$ jointed rods when the $(2n + 1)$ jointed rods for the convex case form an inscribed $(2n + 1)$ -gon. Equally well this expression gives the number of circles with radii $R_1, R_2, \dots, R_{2n+1}$ in which these polygons are inscribed. Further it is easy to show that the series

$$(L) \quad n + (n-1) \binom{2n+1}{1} + (n-2) \binom{2n+2}{2} \dots$$

has exactly this expression for its sum.

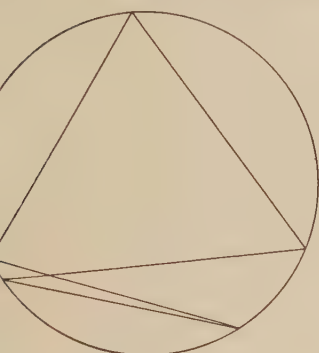


FIG. 1.

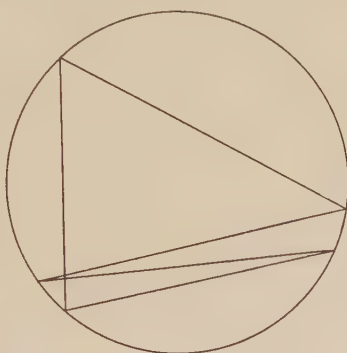


FIG. 2.

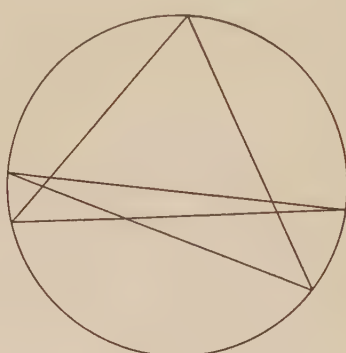


FIG. 3.

Let us examine a few special cases, say when $n = 2, 3, 4$; i. e. when $2n + 1 = 5, 7, 9$. Series (L) gives for these cases the following:

$$(M) \quad n = 2, \quad 2n + 1 = 5, \quad 2 + 5 = 7;$$

$$(N) \quad n = 3, \quad 2n + 1 = 7, \quad 3 + 2(7) + 21 = 38;$$

$$(O) \quad n = 4, \quad 2n + 1 = 9, \quad 4 + 3(9) + 2(36) + 84 = 187.$$

Expressions (M), (N) and (O) give the number of polygons formed by five, seven and nine jointed rods. Equally well these expressions give the number of circles in which these polygons are inscribed. In a similar manner we might extend this to $(2n + 1)$ jointed rods.

Let us consider the numbers given by the last term in the left hand side of the expressions (M), (N) and (O), that is the numbers 5, 21 and 84. These are the number of times that each of the $(2n + 1)$ -gons degenerate into an equilateral triangle when all the sides are equal. In sec. 5 above we saw that in the case of the regular pentagon we obtained two inscribed pentagons, the regular convex and the regular star pentagon, and the degenerate case of the equilateral triangle five times over. For the heptagon the number of degenerate cases is 21, while for the 9-gon it is 84.

Let us examine each of these cases and see how this comes about. The case of the pentagon resolves itself into this: with five rods of equal length

labelled 1, 2, 3, 4, 5 forming a regular convex pentagon, how many distinct equilateral triangles can we form with these five rods? The answer is five. This may be accomplished as follows: bring sides 1 and 2 of the pentagon together until an angle of 60° is formed, these two sides are two of the sides, A and B , of the equilateral triangle. Now fold sides 3, 4 and 5 of the pentagon together forming the third side C of the triangle. This will give one of the triangles. Continue this procedure by taking two adjacent sides of the pentagon to form two sides of the triangle and in this manner we will form five distinct triangles as shown below.

Sides of triangle	A	B	C
Sides of pentagon	1	2	345
	2	3	451
	3	4	512
	4	5	123
	5	1	234

In the above we took the regular convex pentagon but the same results would have been obtained with the regular crossed or star pentagon.

For the heptagon we have the same problem. With seven rods of equal length labelled 1, 2, 3, 4, 5, 6, 7 how many distinct equilateral triangles can be formed? The answer is 21; this agrees with the number given by expression (N) above. They are as follows:

A	B	C
1	2	34567
.	.	.
7	1	23456
123	456	7
.	.	.
712	345	6
1234	56	7
.	.	.
7123	45	6

There are seven distinct triangles in each group which makes a total of twenty-one.

In a similar manner we can obtain eighty-four distinct equilateral triangles with nine labelled rods of equal length. We might extend this process on to polygons of $(2n + 1)$ equal sides where we would obtain $\binom{2n + 1}{n - 1}$ equilateral triangles.

VITA.

The writer, Allie W. Richeson, was born July 3, 1897, near Blantons, Virginia and received his early education in the public schools at that place. He entered the University of Richmond, September 1914, and was graduated with the degree of Bachelor of Science in June 1918. He was instructor of Mathematics at Ogden College during 1919-1920 and since that time he has been teaching Mathematics at the University of Maryland. He entered the Graduate School of the Johns Hopkins University in 1923 and received the degree of Master of Arts in 1925.

To all of his instructors, and especially to Dr. Morley, he desires to express his sincere thanks for the inspiration and benefit gained by following courses under their guidance.

